GENERALIZED CONVEX KERNELS

BY

A. M. BRUCKNER(*) AND J. B. BRUCKNER

ABSTRACT

The notion of the convex kernel of a set D is generalized to that of the n-th order kernel of D. Such kernels are studied for compact, simply connected subsets of the Euclidean plane. In particular, it is shown that under certain circumstances [see Theorem 4 and also section 5], these kernels have rather simple structures.

1. **Introduction.** Horn and Valentine [3] have generalized the notion of convex set to that of L_n set. A set D in the Euclidean plane E_2 is called an L_n set if for every pair of points x and y in D, there is a polygonal line of at most n segments lying in D which joins x to y. Such sets can be used to approximate (in the sense of the Hausdorff metric) any compact, connected set (see [1]). The results of [1] have been extended by McCoy [4] to complete locally compact convex metric spaces. For some work concerning partitions of Euclidean spaces into L_n sets, the reader is referred to Ceder [2].

The convex kernel of a set D is defined to be the set of points x in D such that for all y in D, the segment joining x to y is contained in D. It is well known that the convex kernel of a set is itself convex.

The purpose of this note is to study the nth order kernel of D, by which we mean the set of points x in D such that for each y in D there is a polygonal line of at most n segments lying in D, and joining x to y.

If D is the boundary of a square, for example, the 2nd order kernel consists of the four corners of the square. By considering examples similar to this, one can easily verify that the nth order kernel of a set need not even be connected if $n > 1$. The essential difference between the cases $n = 1$ and $n > 1$ is that there is a unique line determined by any pair of points x and y, but there is an infinitude of polygonal lines with at most n segments joining x to y if $n > 1$. As we shall see, the assumption that D is simply connected partially overcomes this difficulty.

2. Notation and terminology. In the sequel, D will denote a compact, simply connected set in E_2. If B is a set, then δB will denote its boundary and $\sim B$ its complement. We shall use the notation $\langle p_0, p_1, \ldots, p_n \rangle$ for the n-sided polygonal

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line (n-line) joining p_0 to p_n with $p_1, \ldots, p_{n-1}$ consecutive, intermediate vertices It will always be assumed that $\langle p_0, \ldots, p_n\rangle$ has no self intersections. If x and y are points in D, then by $\rho(x, y)$ we shall mean the minimum number of segments that a polygonal line lying in D and joining x to y can have. A full L_n set is an L_n set for which there exist $x, y \in D$ with $\rho(x, y) = n$. If $x \in D$, then $K_x^n = \{y : \rho(x, y) \leqq n\}$. If no confusion can arise, we shall write K_x in place of K_x^n. When considering a sequence $\{x_i\}$ of points of D, we shall, for simplicity of notation, write K_i for K_{x_i}. We shall denote by K^n the nth order kernel of D. Clearly $K^n = \bigcap_{x \in D} K_x^n$.

3. Some properties of K_x. In this section we record four lemmas which we shall use in the next section. Lemma 1, below, will be used frequently in the sequel.

LEMMA 1. *If $x \in D$ and $y, z \in K_x$ and $\langle y, z\rangle \subset D$, then $\langle y, z\rangle \subset K_x$.*

Proof. Let $L_y = \langle x, y_1, \ldots, y_{n-1}, y\rangle$ and $L_z = \langle x, z_1, \ldots, z_{n-1}, z\rangle$ be n-lines in D joining x to y and z respectively. The n-lines L_y and L_z along with $\langle y, z\rangle$ determine a figure P which is the union of a finite number of simple closed polygons (with interiors) some of which may degenerate into segments. Since D is simply connected, $P \subset D$. If $z \in \langle y_{n-1}, y\rangle$ or $y \in \langle z_{n-1}, z\rangle$, the conclusion follows trivially. Likewise the conclusion is immediate if either L_y or L_z intersects $\langle y, z\rangle$ other than at the endpoints. If not, then let $w \in \langle y, z\rangle$. The point w is in one of the polygons P' of P. Each of the vertices $u_1, \ldots, u_r$ of P' is one of the y_i, one of the z_j or a point at which a segment of L_y intersects a segment of L_z. Let $T = \{v \in P' : \langle w, v\rangle \subset P'\}$. It is clear that there exists q, $1 \leqq q \leqq r$, such that $u_q \in T, u_q \neq y, u_q \neq z$. If for some s, $1 \leqq s \leqq n-1$, $u_q = y_s$ then the $(s+1)$-line $\langle x, y_1, \ldots, y_s, w\rangle$ is contained in P and $w \in K_x$. A similar n-line exists in case $u_q = z_s$ for some s. If u_q is a point of intersection of L_y and L_z, then the extension of the segment $\langle u_q, w\rangle$ (into another polygon P'' of P) intersects the boundary of P at a point t. It is easy to verify that there is an n-line lying in P having terminal side $\langle t, w\rangle$.

LEMMA 2. *If $x \in D$, then K_x^n is a compact, simply connected L_{2n} set.*

Proof. The compactness of K_x^n is obvious.

That K_x is an L_{2n} set follows trivially from the fact that any two points of K_x^n can be joined by a $2n$-line whose middle vertex is x.

It remains to show that $\sim K_x^n$ has no bounded components. Assume $\sim K_x^n$ has a bounded component C. Let $y \in C$ and let $\langle y_1, y_2\rangle$ be a segment through y where $y_1, y_2 \in \delta C$, and there are no other points of δC on $\langle y_1, y\rangle$ and $\langle y, y_2\rangle$. Since boundary points of C must also be boundary points of K_x^n, and since K_x^n is closed, it follows that y_1 and y_2 are in K_x^n. Now each point z of $\langle y_1, y_2\rangle$ is in D, otherwise the component of $\sim D$ containing z would be bounded. By Lemma 1, $y \in K_x^n$ contradicting the hypothesis $y \in C$. Hence K_x^n is simply connected.

LEMMA 3. *Let $x_1, x_2, \ldots, x_m$ be points of D and let $M = K_1 \cap K_2 \cap \ldots \cap K_{m-1}$. Then $\sim(M \cup K_m)$ has no bounded components.*

Proof. Suppose that $\sim(M \cup K_m)$ has a bounded component C. Since C is a bounded, open, connected set, it can be shown that there exist three points $y_1, y_2, y_3 \in \delta C$ such that the open segments (y_1, y_2), (y_2, y_3) and (y_1, y_3) are contained in C. Now since $\delta C \subset M \cup K_m$, two of these three points, say y_1 and y_2, are in M or in K_m. In either case, it follows from Lemma 1 that the entire segment $\langle y_1, y_2 \rangle$ is in M or in K_m. This contradicts the existence of a bounded component of $\sim(M \cup K_m)$.

LEMMA 4. *Let $x_1, \ldots, x_m$ be points of D. Then $K_1^n \cap K_2^n \cap \cdots K_m^n$ is a compact, simply connected L_{2n} set.*

Proof. That $K_1^n \cap \ldots \cap K_m^n$ is compact follows trivially from the compactness of $K_1^n, \ldots, K_m^n$. If $m = 1$, the theorem reduces to Lemma 2. Let $M = K_1^n \cap \cdots \cap K_{m-1}^n$. Assume M is a simply connected, L_{2n} set. Let y, z be points of $M \cap K_m^n$; let $L = \langle y, y_{n-1}, \ldots, y_1, x_m, z_1, \ldots, z_{n-1}, z \rangle$ and $L_M = \langle y, v_1, v_2, \cdots, v_{2n-1}, z \rangle$ be $2n$-lines joining y and z lying in K_m^n and M respectively. These lines determine a figure P which is a union of a finite number of simple closed polygons with interiors some of which may degenerate into segments. We may further assume that L_M is simple and no line segment joining nonadjacent vertices of L_M lies entirely in P. Since $\sim(M \cup K_m^n)$ has no bounded components, $P \subset M \cup K_m^n$. We shall show that there is a polygonal path in $K_m^n \cap M$ having at most the number of segments of L_M.

For each $v \in L_M$ let

$$s(v) = \sup \{ t \in L_M : \langle v, t \rangle \subset P \text{ and } (v, t) \cap \langle z_{n-1}, z \rangle \text{ is empty} \}$$

where the supremum is taken with respect to the natural ordering of L_M from y to z. We shall denote this ordering by "$\prec$".

We first show that if $v \in K_m$, then $s(v) \in K_m^n$. Now, if $s(v) \in \langle z_{n-1}, z \rangle$, the conclusion is trivially true. If $s(v) \notin \langle z_{n-1}, z \rangle$, then for some point p on L, the points v $s(v)$ and p are collinear. If $p \in \langle z_{n-1}, z \rangle$, then $s(v) \in \langle v, p \rangle$ and by Lemma 1, $s(v) \in K_m^n$. If $p \in \langle z_m, z_{m+1} \rangle$, $m \neq n-1$, then the polygonal line $\langle x_m, z_1, \ldots, z_m, p, s(v) \rangle$ lies in D and has at most n segments so that $s(v) \in K_m^n$. A similar polygonal path exists if $p \in \langle y_{m+1}, y_m \rangle$ ($m \neq n-1$ unless $p = y_{n-1}$).

It is clear that if $v_i \preceq v \prec v_{i+1}$ and $(v_i, v_{i+1}) \cap \langle z_{n-1}, z \rangle$ is empty, then $v_{i+1} \preceq s(v)$. If $s(v) \in \langle v_i, v_{i+1} \rangle \cap \langle z_{n-1}, z \rangle$, then the line $\langle v, s(v), z \rangle$ has at most as many segments (two or, in the degenerate case $s(v) = z$, one) as $\langle v, v_{i+1}, \ldots, v_{2n-1}, z \rangle$. Thus the polygonal line

$$\langle y, s(y), s^2(y), \ldots, z \rangle$$

has at most $2n$ segments. It follows from Lemma 1 that this $2n$-line lies in $K_m^n \cap M$.

The simple connectedness of $K_m^n \cap M$ now follows from the simple connectedness of $K_1, \ldots, K_m$.

4. The nth order kernel. We now proceed to study the nth order kernel of D. In Theorems 1 and 3 we obtain representations for K^n. Theorem 2 shows that K^n shares some of the properties of the sets K_x obtained in Lemma 2. In Theorem 4 we show that for a full L_{2n} set, K^n is itself an L_2 set.

THEOREM 1. *Let D' be a dense subset of D. Then $K^n = \bigcap_{x \in D'} K_x^n$.*

Proof. Let $R = \bigcap_{x \in D'} K_x^n$. Trivially, $K^n \subset R$. Now assume $y \in R$. Then $y \in K_x^n$ for each $x \in D'$. Equivalently $D' \subset K_y^n$. But K_y^n is closed so that K_y^n contains the closure of D' which is D. Thus $D \subset K_y^n$ and $y \in K^n$ so that $R \subset K^n$.

THEOREM 2. *The* nth *order kernel is a compact, simply xconnected, L_{2n} set.*

Proof. Let $D' = \{x_1, \ldots, x_m, \ldots\}$ be a countable, dense subset of D; let $M_m = K_1^n \cap \ldots \cap K_m^n$. The sets M_m form a decreasing sequence of compact, L_{2n} sets whose intersection K^n is therefore a compact, L_{2n} set (see [1]: Theorem 2). The simple connectedness of K follows immediately from the connectedness of K^n and the simple connectedness of $K_1^n, K_2^n, \ldots$.

THEOREM 3. *Let $S = \cap K_x^n$, the intersection being taken over all points $x \in \delta D$. Then $S = K^n$.*

Proof. Trivially $K^n \subset S$. Let $x \in D \sim K^n$ and let $y \in \sim K_x^n$. If $y \in \delta D$ then $x \in D \sim S$. If y is an interior point of D, then the component of $\sim K_x^n$ containing y must contain a boundary point of D, for otherwise K_x^n would not be simply connected. Thus $x \notin S$ and $S \subset K^n$.

The corollary below generalizes [3 : Theorem 1.4].

COROLLARY. *If $y \in K_x^n$ for all $x, y \in \delta D$ then D is an L_n set.*

Proof. If, for every $x \in \delta D$, $y \in K_x^n$ for all $y \in \delta D$, then by Theorem 3, $\delta D \subset K^n$ Since K^n is simply connected, it follows that $D \subset K^n$. Thus for each $x, y \in D$ we have $\rho(x,y) \leqq n$.

LEMMA 5. *Let D be a full L_{2n} set, let $x, y \in K^n$ and let α be a point of D such that $\rho(\alpha, x) = \rho(\alpha, y) = n$. Then $K_x^1 \cap K_y^1 \cap K_\alpha^n$ is non empty.*

Proof. We first show that $K_x^1 \cap K_y^1$ is non empty. Since D is a full L_{2n} set, there are points β and γ with $\rho(\beta,\gamma) = 2n$. Let $L_y = \langle \beta, b_{n-1}, \ldots, b_1, y, c_1, \ldots, c_{n-1}, \gamma \rangle$ and $L_x = \langle \beta, \beta_{n-1}, \ldots, \beta_1, x, \gamma_1, \ldots, \gamma_{n-1}, \gamma \rangle$ be $2n$-lines joining β and γ via x and y respectively. The lines L_x and L_y determine a figure P which is the union of a finite number of simple closed polygons with interiors, some of which may degenerate into segments.

If $x \in L_y$ (or $y \in L_x$) then $\langle x, y \rangle \subset L_y (\langle x, y \rangle \subset L_x)$, for otherwise we would have $\rho(\beta, \gamma) < 2n$; hence $\langle x, y \rangle \subset K_x^1 \cap K_y^1$.

If $x \notin L_y$ and $y \notin L_x$ then x and y are accessible to the interior of P. There must be a segment $\langle x, v\rangle$ lying in P with $v \in L_y$ since $\rho(\beta,\gamma) = 2n$. Similarly there is a segment $\langle y,t\rangle \subset P$ with $t \in L_x$. It is immediately clear that

$$t \in \langle \beta_2, \beta_1, x, \gamma_1, \gamma_2\rangle$$

and

$$v \in \langle b_2, b_1, y, c_1, c_2\rangle .$$

Several situations can occur.

If $t \in \langle \beta_1, x, \gamma_1\rangle$ (or $v \in \langle b_1, y, c_1\rangle$) then $v \in K_x^1 \cap K_y^1$) ($t \in K_x^1 \cap K_y^1$).

If $t \in \langle \beta_2, \beta_1\rangle$ and $v \in \langle b_2, b_1\rangle$ (or $t \in \langle \gamma_1, \gamma_2\rangle$ and $v \in \langle c_1, c_2\rangle$) then it can be verified that $\langle x,v\rangle \cap \langle y,t\rangle = \{p\} \in P$ and $p \in K_x^1 \cap K_y^1$.

If $t \in \langle \beta_2, \beta_1\rangle$ and $v \in \langle c_1, c_2\rangle$ (or $t \in \langle \gamma_1, \gamma_2\rangle$ and $v \in \langle b_2, b_1\rangle$) then it can be verified that one of the vertices of the hexagon $\langle y, t, \beta_1, x, v, c_1, y\rangle$ ($\langle y, t, \gamma_1, x, v, b_1, y\rangle$) is in $K_x^1 \cap K_y^1$.

We proceed to show that $K_x^1 \cap K_y^1 \cap K_\alpha^n$ is non empty. Let p be a point of $K_x^1 \cap K_y^1$. The lines $L_1 = \langle x, a_{n-1}, \dots, a_1, \alpha, \alpha_1, \dots, \alpha_{n-1}, y\rangle$ and $L_2 = \langle x, p, y\rangle$ determine a figure P^*. An analysis of P^* similar to the above analysis of P, but using the fact that any polygonal path between x and α or y and α must have at least n segments, verifies that there is a point $v \in P^*$ which lies in $K_x^1 \cap K_y^1 \cap K_\alpha^n$.

THEOREM 4. *Let D be a full L_{2n} set. Then K^n is an L_2 set.*

Proof. If K^n is not an L_2 set, then there are points $x, y \in K^n$ such that for each $t \in K_x^1 \cap K_y^1$, there exists $\alpha(t) \in D$ with $t \in \sim K_{\alpha(t)}^n$. Since $K_x^1 \cap K_y^1$ is compact, there exist points $\alpha_1, \alpha_2, \dots, \alpha_p \in D$ such that for each $t \in K_x^1 \cap K_y^1$, $t \in \sim K_i^n$ for some i, $1 \leqq i \leqq p$. Each of the α_i satisfies $\rho(\alpha_i, x) = \rho(\alpha_i, y) = n$.

We will show that $K_1^n \cap K_2^n \cap \dots \cap K_q^n \cap K_x^1 \cap K_y^1$ is not empty for $q = 1, 2, \dots$. Now it follows from Lemma 5, that for each α_i there is a point $t_i \in K_x^1 \cap K_y^1 \cap K_i^n$. Hence the conclusion is valid for $q = 1$. Assume the conclusion holds for $q-1$. Then there exists $t \in K_1^n \cap \dots \cap K_{q-1}^n \cap K_x^1 \cap K_y^1$. By Lemma 5 we have the existence of t' in $K_q^n \cap K_x^1 \cap K_y^1$. The 2-lines $L = \langle x, t, y\rangle$ and $L' = \langle x, t', y\rangle$ determine a figure P which is the union of at most two simple closed polygons with interior, some of which may degenarete into segments. Since D is simply connected, $P \subset D$. Now $\langle x, \rangle$ is not contained in P. For then we would have $\langle x, y\rangle \subset K^n$ contrary to the assumption that x and y cannot be joined by a 2-line in K^n.

If L and L' intersect at a point v, $v \neq x, y$, then $v \in K_1^n \cap \dots \cap K_q^n \cap K_x^1 \cap K_y^1$.

If $L \cap L' = \{x\} \cup \{y\}$ then for one of the lines L (or L') we have $\langle x, t(\text{or } t'), y\rangle \sim \{x\} \sim \{y\} \subset \operatorname{Int} P^*$ where P^* is the figure determined by $\langle x, t'(\text{or } t), y\rangle$ and $\langle x, y\rangle$. It is easily verified that $t(\text{or } t') \in K_1^n \cap \dots \cap K_q^n \cap K_x^1 \cap K_y^1$. By induction there is a point $v \in K_1^n \cap \dots \cap K_p^n \cap K_x^1 \cap K_y^1$ contrary to the assumption that the sets $\sim K_i^n$ cover $K_x^1 \cap K_y^1$.

5. Kernels of nowhere dense sets. Throughout this section D is nowhere dense

in addition to being compact and simply connected. The following result is easy to verify and the proof is ommitted.

LEMMA 6. *Let* $x, y \in D$. *If* L_p *and* L_q *are* p- *and* q-*lines respectively*, $p \leqq q$, *joining* x *to* y *and* $L = \{t: t \in L_p \cap L_q\}$ *then* L *is an* r-*line joining* x *to* y *with* $r \leqq p$.

The following theorems can be proved by applications of Lemma 6.

THEOREM 5. *Let* D *be a full* L_{2n} $[L_{2n-1}]$ *set; let* $x, y \in D$ *with* $\rho(x,y) = 2_n$ $[\rho(x,y) = 2n - 1]$. If $\langle x, \alpha_1, \ldots, \alpha_n, \ldots, \alpha_{2n-1}, y\rangle$ $[\langle x, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n, \ldots, \alpha_{2n-2}, y\rangle$ *is a* $2n$-*line* $[(2n-1)$-*line*$]$ *joining* x *to* y *then* $\alpha_n \in K^n$ $[\langle \alpha_{n-1}, \alpha_n\rangle \subset K^n]$.

THEOREM 6. *Let* D *be a full* $L_{2n}[L_{2n-1}]$ *set. Then* K^n *is a single point* $[K^n$ *is a single segment*$]$.

THEOREM 7. *Let* D *be a full* $L_{2n}[L_{2n-1}]$ *set and for* $p > n$, *let* K^p *denote its pth order kernel; let* $p = n + q$. *Then* K^p *is an* $L_{2q}[L_{2q+1}]$ *set.*

A basic difference between the cases in which D is nowhere dense and the general case is that in the former, any two points of D determine a unique path of fewest segments (as Lemma 6 illustrates), whereas this is not so in the general case. Theorems 5 and 6 obviously have no counterparts in the general case. We suspect that Theorem 7 does have an analogue in the general case but have been unable to prove this. It is worth noting that Theorem 6 implies that the nth order kernel of a full $L_{2n}[L_{2n-1}]$ set is non empty in case D is nowhere dense. This is not necessarily true in the general case. For example, if D is the simply connected set determined by a triangle whose sides are extended one unit in each direction, then D is a full L_2 set with an empty first order (convex) kernel.

6. Some conclusing remarks. We conclude with several observations. Simple examples show that K^n might be contained entirely in the interior of D or entirely in δD, even if D is bounded by a simple Jordan curve.

It can be shown that if $x \in D$ then the boundary of a component of $D \sim K_x^n$ can be decomposed into two sets A and B, where $A \subset \delta D$ and B is either a subinterval of a single segment of an n-line in K_x^n or empty. No corresponding statement can be made for the boundary of a component of $D \sim K^n$.

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UNIVERSITY OF CALIFORNIA, SANTA BARBARA
AND SANTA BARBARA, CALIFORNIA